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Some infinite divisibility properties of the reciprocal of planar Brownian motion exit time from a cone

S. Vakeroudis ^{*†} M. Yor ^{*‡}

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Abstract

With the help of the Gauss-Laplace transform for the exit time from a cone of planar Brownian motion, we obtain some infinite divisibility properties for the reciprocal of this exit time.

Key words: Bougerol's identity, infinite divisibility, Chebyshev polynomials, Lévy measure, Thorin measure, generalized Gamma convolution (GGC).

MSC Classification (2010): 60J65, 60E07.

1 Introduction

Let $(Z_t = X_t + iY_t, t \geq 0)$ denote a standard planar Brownian motion[§], starting from $x_0 + i0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively from x_0 and 0.

It is well known [ItMK65] that, since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. Hence, the continuous winding process $\theta_t = \text{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined. Using a scaling argument, we may assume $x_0 = 1$, without loss of generality, since, with obvious notation:

$$\left(Z_t^{(x_0)}, t \geq 0 \right) \stackrel{(law)}{=} \left(x_0 Z_{(t/x_0^2)}^{(1)}, t \geq 0 \right). \quad (1)$$

^{*}Laboratoire de Probabilités et Modèles Aléatoires (LPMA) CNRS : UMR7599, Université Pierre et Marie Curie - Paris VI, Université Paris-Diderot - Paris VII, 4 Place Jussieu, 75252 Paris Cedex 05, France. E-mail: stavros.vakeroudis@etu.upmc.fr

[†]Probability and Statistics Group, School of Mathematics, University of Manchester, Alan Turing Building, Oxford Road, Manchester M13 9PL, United Kingdom.

[‡]Institut Universitaire de France, Paris, France. E-mail: yormarc@aol.com

[§]When we write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

From now on, we shall take $x_0 = 1$.

Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}}, \quad (2)$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log 1 + i0 = 0$ (for further study of the Bessel clock H , see [Yor80]).

We may rewrite (2) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}. \quad (3)$$

One now easily obtains that the two σ -fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

Bougerol's celebrated identity in law ([Bou83, ADY97] and [Yor01] (p. 200)), which says that:

$$\text{for fixed } t, \quad \sinh(\beta_t) \stackrel{(law)}{=} \delta_{A_t(\beta)} \quad (4)$$

where $(\beta_u, u \geq 0)$ is 1-dimensional BM, $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$ and $(\delta_v, v \geq 0)$ is another BM, independent of $(\beta_u, u \geq 0)$, will also be used. We define the random times $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$, and $T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$, ($c > 0$). From the skew-product representation (3) of planar Brownian motion, we obtain [ReY99]:

$$A_{T_c^{|\gamma|}}(\beta) \equiv \int_0^{T_c^{|\gamma|}} ds \exp(2\beta_s) = H_u^{-1} \Big|_{u=T_c^{|\gamma|}} = T_c^{|\theta|}. \quad (5)$$

Then, Bougerol's identity (4) for the random time $T_c^{|\theta|}$ yields the following [Vak11, VaY11]:

Proposition 1.1 *The distribution of $T_c^{|\theta|}$ is characterized by:*

$$E \left[\sqrt{\frac{2c^2}{\pi T_c^{|\theta|}}} \exp \left(-\frac{x}{2T_c^{|\theta|}} \right) \right] = \frac{1}{\sqrt{1+x}} \varphi_m(x), \quad (6)$$

for every $x \geq 0$, with $m = \frac{\pi}{2c}$, and

$$\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}, \quad \text{with } G_{\pm}(x) = \sqrt{1+x} \pm \sqrt{x}. \quad (7)$$

Comment and Terminology:

If $S > 0$ a.s. is independent from a Brownian motion $(\delta_u, u \geq 0)$, we call the density of δ_S , which is:

$$E \left[\frac{1}{\sqrt{2\pi S}} \exp \left(-\frac{x^2}{2S} \right) \right] \quad (8)$$

the Gauss-Laplace transform of S (see e.g. [ChY03] ex.4.16, or [BPY01]). Thus, formula (6) expresses - up to simple changes - the Gauss-Laplace transform of $T_c^{|\theta|}$.

We also recall several notions which will be used throughout the following text:

a) A stochastic process $\zeta = (\zeta_t, t \geq 0)$ is called a *Lévy process* if $\zeta_0 = 0$ a.s., it has stationary and independent increments and it is almost surely right continuous with left limits. A Lévy process which is increasing is called a *subordinator*.

b) Following e.g. [ReY99], a probability measure π on \mathbb{R} (resp. a real-valued random variable with law π) is said to be *infinitely divisible* if, for any $n \geq 1$, there is a probability measure π_n such that $\pi = \pi_n^{*n}$ (resp. if ζ_1, \dots, ζ_n are n i.i.d. random variables, $\zeta \stackrel{(law)}{=} \zeta_1 + \dots + \zeta_n$). For instance, Gaussian, Poisson and Cauchy variables are infinitely divisible.

It is well-known that (e.g. [Ber96]), π is infinitely divisible if and only if, its Fourier transform $\hat{\pi}$ is equal to $\exp(\psi)$, with:

$$\psi(u) = ibu - \frac{\sigma^2 u^2}{2} + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \nu(dx),$$

where $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a Radon measure on $\mathbb{R} \setminus \{0\}$ such that:

$$\int \frac{x^2}{1+x^2} \nu(dx) < \infty.$$

This expression of $\hat{\pi}$ is known as the *Lévy-Khintchine formula* and the measure ν as the *Lévy measure*.

c) Following [Bon92] (p.29) and [JRY08], a positive random variable Γ is a *generalized Gamma convolution* (GGC) if there exists a positive Radon measure μ on $]0, \infty[$ such that:

$$E[e^{-\lambda\Gamma}] = \exp \left(- \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x} \int_0^\infty e^{-xz} \mu(dz) \right) \quad (9)$$

$$= \exp \left(- \int_0^\infty \log \left(1 + \frac{\lambda}{z} \right) \mu(dz) \right), \quad (10)$$

with:

$$\int_{]0,1]} |\log x| \mu(dx) \quad \text{and} \quad \int_{[1,\infty[} \frac{\mu(dx)}{x} < \infty. \quad (11)$$

We remark that (10) follows immediately from (9) using the elementary Frullani formula (see e.g. [Leb72], p.6). The measure μ is called *Thorin's measure* associated with Γ .

We return now to the case of planar Brownian motion and the exit times from a cone. Below, we state and prove the following:

Proposition 1.2 *For every integer m , the function $x \rightarrow \varphi_m(x)$, is the Laplace transform of an infinitely divisible random variable K ; more specifically, the following decompositions hold:*

- for $m = 2n + 1$,

$$K = \frac{\mathcal{N}^2}{2} + \sum_{k=1}^n a_k \mathbf{e}_k, \quad a_k = \frac{1}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right)}; \quad k = 1, 2, \dots, n, \quad (12)$$

- for $m = 2n$,

$$K = \sum_{k=1}^n b_k \mathbf{e}_k, \quad b_k = \frac{1}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n}\right)}; \quad k = 1, 2, \dots, n, \quad (13)$$

where \mathcal{N} is a centered, reduced Gaussian variable and \mathbf{e}_k , $k \leq n$ are n independent exponential variables, with expectation 1.

Looking at formula (6), it is also natural to consider:

$$\tilde{\varphi}_m(x) \equiv \frac{1}{\sqrt{1+x}} \varphi_m(x). \quad (14)$$

We note that:

$$\tilde{K} \equiv \frac{\mathcal{N}^2}{2} + K, \quad (15)$$

admits the RHS of (6) as its Laplace transform. Hence,

- for $m = 2n + 1$,

$$\tilde{K} \stackrel{(law)}{=} \mathbf{e}_0 + \sum_{k=1}^n a_k \mathbf{e}_k, \quad (16)$$

- for $m = 2n$,

$$\tilde{K} \stackrel{(law)}{=} \frac{\mathcal{N}^2}{2} + \sum_{k=1}^n b_k \mathbf{e}_k, \quad (17)$$

with obvious notation.

In Section 2 we first illustrate Proposition 1.2 for $m = 1$ and $m = 2$; we may also verify equation (6) by using the laws of $T_c^{|\theta|}$, for $c = \pi/2$ and $c = \pi/4$, which are well known [ReY99].

In Section 3, we prove Proposition 1.2, where the Chebyshev polynomials play an essential role, we calculate the Lévy measure in the Lévy-Khintchine representation of φ_m and we obtain the following asymptotic result:

Proposition 1.3 *With c denoting a positive constant, the distribution of $T_{c\varepsilon}^{|\theta|}$, for every $x \geq 0$, follows the asymptotics:*

$$\left(E \left[\sqrt{\frac{2(c\varepsilon)^2}{\pi T_{c\varepsilon}^{|\theta|}}} \exp \left(-\frac{x}{2T_{c\varepsilon}^{|\theta|}} \right) \right] \right)^{1/\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{(\sqrt{x} + \sqrt{1+x})^{\pi/2c}}, \quad (18)$$

which, from [JRY08], is the Laplace transform of a subordinator $(\Gamma_t(\mathbb{G}_{1/2}), t \geq 0)$ with Thorin measure that of the arc sine law, taken at $t = \pi/2c$.

Finally, we state a conjecture concerning the case where m is not necessarily an integer.

2 Examples

2.1 $\boxed{m = 1 \Rightarrow c = \frac{\pi}{2}}$

Then:

$$\tilde{\varphi}_1(x) = \frac{1}{1+x}, \quad (19)$$

is the Laplace transform of an exponential variable \mathbf{e}_1 .

Indeed, with $(Z_t = X_t + iY_t = |Z_t| \exp(i\theta_t), t \geq 0)$ a planar BM starting from $(1, 0)$, $T_{\pi/2}^{|\theta|} = \inf\{t : X_t = 0\} = \inf\{t : X_t^0 = 1\}$,

with $(X_t^0, t \geq 0)$ denoting another one-dimensional BM starting from 0. Formula (6) states that:

$$E \left[\sqrt{\frac{2}{\pi T_{\pi/2}^{|\theta|}}} \exp \left(-\frac{x}{2T_{\pi/2}^{|\theta|}} \right) \right] = \frac{1}{1+x}. \quad (20)$$

However, we know that: $T_{\pi/2}^{|\theta|} \stackrel{(law)}{=} \frac{1}{N^2}$, $N \sim \mathcal{N}(0, 1)$.

The LHS of the previous equality (20) gives:

$$E \left[\sqrt{\frac{2}{\pi}} |N| \exp \left(-\frac{x}{2} N^2 \right) \right] = \int_0^\infty dy y e^{-\frac{x+1}{2} y^2} = \frac{1}{1+x}, \quad (21)$$

thus, we have verified directly that (20) holds.

2.2 $\boxed{m = 2 \Rightarrow c = \frac{\pi}{4}}$

Similarly,

$$\tilde{\varphi}_2(x) = \frac{1}{\sqrt{1+x}} \frac{1}{1+2x}, \quad (22)$$

is the Laplace transform of the variable $\frac{N^2}{2} + 2\mathbf{e}_1$.

Again, this can be shown directly; indeed, with obvious notation:

$$\begin{aligned} T_{\pi/4}^{|\theta|} &= \inf\{t : X_t + Y_t = 0, \text{ or } X_t - Y_t = 0\} \\ &= \inf\left\{t : \frac{X_t^0 + Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}, \text{ or } \frac{X_t^0 - Y_t}{\sqrt{2}} = \frac{1}{\sqrt{2}}\right\} \\ &= T_{1/\sqrt{2}} \wedge \tilde{T}_{1/\sqrt{2}} \stackrel{(law)}{=} \frac{1}{2} (T \wedge \tilde{T}). \end{aligned}$$

Hence, formula (6) now writes, in this particular case:

$$E \left[\sqrt{\frac{\pi}{4(T \wedge \tilde{T})}} \exp \left(-\frac{x}{T \wedge \tilde{T}} \right) \right] = \frac{1}{\sqrt{1+x}} \frac{1}{1+2x}. \quad (23)$$

This is easily proven, using: $T \stackrel{(law)}{=} \frac{1}{N^2}, \tilde{T} \stackrel{(law)}{=} \frac{1}{\tilde{N}^2}$, which yields:

$$\begin{aligned} E \left[\left(|N| \vee |\tilde{N}| \right) \exp \left(-x \left(N^2 \vee \tilde{N}^2 \right) \right) \right] &= 2E \left[|N| \exp \left(-xN^2 \right) 1_{(|N| \geq |\tilde{N}|)} \right] \\ &= C \int_0^\infty du \, u \, e^{-xu^2} e^{-\frac{u^2}{2}} \int_0^u dy \, e^{-\frac{y^2}{2}}. \end{aligned}$$

Fubini's theorem now implies that (23) holds.

Remark 2.1 *In a first draft, we continued looking at the cases: $m = 3, 4, 5, 6, \dots$, in a direct manner. But, these studies are now superseded by the general discussion in Section 3.*

2.3 A "small" generalization

As we just wrote in Remark 2.1, before finding the proof of Proposition 1.2 (see below, Subsection 3.1), we kept developing examples for larger values of m , and in particular, we encountered quantities of the form:

$$\frac{1}{\mathcal{P}_{u,v}(x)}, \text{ with } \mathcal{P}_{u,v}(x) = 1 + ux + vx^2. \quad (24)$$

These quantities turn out to be the Laplace transforms of variables of the form $a\mathbf{e} + b\mathbf{e}'$, with $a, b > 0$ constants and \mathbf{e}, \mathbf{e}' two independent exponential variables. In this Subsection, we characterize the polynomials $\mathcal{P}_{u,v}(x)$ such that this is so.

Lemma 2.2 *a) A necessary and sufficient condition for $1/\mathcal{P}_{u,v}$ to be the Laplace transform of the law of $a\mathbf{e} + b\mathbf{e}'$, is:*

$$u, v > 0 \quad \text{and} \quad \Delta \equiv u^2 - 4v \geq 0. \quad (25)$$

b) Then, we obtain:

$$a = \frac{u - \sqrt{\Delta}}{2} \quad ; \quad b = \frac{u + \sqrt{\Delta}}{2}. \quad (26)$$

Proof of Lemma 2.2

i) $1/\mathcal{P}_{u,v}$ is the Laplace transform of $a\mathbf{e} + b\mathbf{e}'$, then:

$$\mathcal{P}_{u,v}(x) = (1 + ax)(1 + bx).$$

Both $u = a + b$ and $v = ab$ are positive.

Moreover, $\mathcal{P}_{u,v}$ admits two real roots, thus $\Delta \equiv u^2 - 4v \geq 0$; i.e.: (25) is satisfied.

ii) Conversely, if the two conditions (25) are satisfied, then the 2 roots of the polynomial are $-1/a$ and $-1/b$. Hence, $\mathcal{P}_{u,v}(x) = C(1 + ax)(1 + bx)$, where C is a constant. However, from the definition of $\mathcal{P}_{u,v}$ (24), we have: $\mathcal{P}_{u,v}(0) = 1$, hence $C = 1$. Thus, $1/\mathcal{P}_{u,v}$ is the Laplace transform of $a\mathbf{e} + b\mathbf{e}'$.

iii) To show b), we note that:

$$\left\{-\frac{1}{a}, -\frac{1}{b}\right\} = \left\{\frac{-u - \sqrt{\Delta}}{2v}, \frac{-u + \sqrt{\Delta}}{2v}\right\}.$$

as well as: $(u - \sqrt{\Delta})(u + \sqrt{\Delta}) = 4v$, which finishes the proof of the second part of the Lemma. ■

3 A discussion of Proposition 1.2 in terms of the Chebyshev polynomials

3.1 Proof of Proposition 1.2

a) Assuming, to begin with, the validity of our Proposition 1.2, for any integer m , the function φ_m should admit the following representation:

$$\varphi_m(x) = \frac{1}{D_m(x)}, \quad (27)$$

where

- for $m = 2n + 1$, $D_m(x) = \sqrt{1+x}P_n(x)$, with $P_n(x) = \prod_{k=1}^n (1 + a_k x)$,
- for $m = 2n$, $D_m(x) = Q_n(x)$, with $Q_n(x) = \prod_{k=1}^n (1 + b_k x)$.

In particular, P_n and Q_n are polynomials of degree n , each of which has its n zeros, that is $(-1/a_k; k = 1, 2, \dots, n)$, resp. $(-1/b_k; k = 1, 2, \dots, n)$, on the negative axis \mathbb{R}_- .

It is not difficult, from the explicit expression of $D_m(x) = \frac{1}{2}((G_+(x))^m + (G_-(x))^m)$, to find the polynomials P_n and Q_n . They are given by the formulas:

$$\begin{cases} P_n(x) = \sum_{k=0}^n C_{2n+1}^{2k+1} (1+x)^k x^{n-k}, \\ Q_n(x) = \sum_{k=0}^n C_{2n}^{2k} (1+x)^k x^{n-k}. \end{cases} \quad (28)$$

In order to prove Proposition 1.2, we shall make use of Chebyshev's polynomials of the first kind (see e.g. [Riv90] ex.1.1.1 p.5 or [KVA02] ex.25, p.195):

$$\begin{aligned} T_m(y) &\equiv \frac{(y + \sqrt{y^2 - 1})^m + (y - \sqrt{y^2 - 1})^m}{2} \\ &\equiv \begin{cases} \cos(m \arg \cos(y)), & y \in [-1, 1] \\ \cosh(m \arg \cosh(y)), & y \geq 1 \\ (-1)^m \cosh(m \arg \cosh(-y)), & y \leq -1. \end{cases} \end{aligned} \quad (29)$$

b) We now start the proof of Proposition 1.2 in earnest. First, we remark that:

$$\varphi_m(x) = \frac{1}{T_m(\sqrt{1+x})}, \quad (30)$$

hence:

$$D_m(x) = T_m\left(\sqrt{1+x}\right), \quad (31)$$

with $x \geq -1$, thus we are interested only in the positive zeros of T_m , and we study separately the cases m odd and m even.

$$\boxed{m = 2n + 1}$$

$$D_{2n+1}(y) \equiv \sqrt{1+y}P_n(y) = T_{2n+1}\left(\sqrt{1+y}\right)$$

and the zeros of T_{2n+1} are: $x_k = \cos\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right)$, $k = 1, 2, \dots, (2n+1)$. However, x_k is positive if and only if $k = 1, 2, \dots, n$, thus:

$$y_k = x_k^2 - 1 = \cos^2\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right) - 1 = -\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right); k = 1, 2, \dots, n.$$

Finally:

$$a_k = \frac{1}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right)}; k = 1, 2, \dots, n, \quad (32)$$

and

$$P_n(x) = \prod_{k=1}^n \left(1 + \frac{x}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n+1}\right)}\right). \quad (33)$$

$\boxed{m = 2n}$ Similarly, we obtain:

$$b_k = \frac{1}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n}\right)}; k = 1, 2, \dots, n, \quad (34)$$

and

$$Q_n(x) = \prod_{k=1}^n \left(1 + \frac{x}{\sin^2\left(\frac{\pi}{2} \frac{2k-1}{2n}\right)}\right). \quad (35)$$

■

3.2 Search for the Lévy measure of φ_m and proof of Proposition 1.3

We have proved that φ_m is infinitely divisible. In this Subsection, we shall calculate its Lévy measure. For this purpose, we shall make use of the following (recall that \mathbf{e}_k , $k \leq n$ are n independent exponential variables, with expectation 1):

Lemma 3.1 *With $(c_k, k = 1, 2, \dots, n)$ denoting a sequence of positive constants, $\prod_{k=1}^n \frac{1}{(1+c_k x)}$ is the Laplace transform of $\sum_{k=1}^n c_k \mathbf{e}_k$, which is an infinitely divisible random variable with Lévy measure:*

$$\frac{dz}{z} \sum_{k=1}^n e^{-z/c_k}.$$

Proof of Lemma 3.1

Using the elementary Frullani formula (see e.g. [Leb72], p.6), we have:

$$\begin{aligned} \prod_{k=1}^n \frac{1}{(1+c_k x)} &= \exp \left\{ - \sum_{k=1}^n \log(1+c_k x) \right\} = \exp \left\{ - \sum_{k=1}^n \int_0^\infty \frac{dy}{y} e^{-y} (1 - e^{-c_k x y}) \right\} \\ &\stackrel{z=c_k y}{=} \exp \left\{ - \sum_{k=1}^n \int_0^\infty \frac{dz}{z} e^{-z/c_k} (1 - e^{-x z}) \right\}, \end{aligned}$$

which finishes the proof. ■

We return now to the proof of Proposition 1.3 and we study separately the cases m odd and m even and we apply Lemma 3.1 with $c_k = a_k$ and $c_k = b_k$ respectively.

$m = 2n + 1$ Lemma 3.1 yields that, $\prod_{k=1}^n \frac{1}{(1+a_k x)}$ is the Laplace transform of an infinitely divisible random variable with Lévy measure:

$$\nu_+(dz) = \frac{dz}{z} \sum_{k=1}^n e^{-z/a_k}. \quad (36)$$

Moreover:

$$\frac{1}{(\prod_{k=1}^n (1+a_k x))^{1/n}} = \exp \left\{ - \int_0^\infty \frac{dz}{z} (1 - e^{-xz}) \frac{1}{n} \sum_{k=1}^n \exp \left\{ -\frac{z}{a_k} \right\} \right\}, \quad (37)$$

and $\frac{1}{(\prod_{k=1}^n (1+a_k x))^{1/n}}$, for $n \rightarrow \infty$, converges to the Laplace transform of a variable which is a generalized Gamma convolution (GGC) with Thorin measure density:

$$\begin{aligned} \mu_+(z) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \exp \left\{ -\frac{z}{a_k} \right\} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \exp \left\{ -z \sin^2 \left(\frac{\pi}{2} \frac{2k-1}{2n+1} \right) \right\} \\ &= \int_0^1 du \exp \left\{ -z \sin^2 \left(\frac{\pi}{2} u \right) \right\} \stackrel{v=\frac{\pi}{2}u}{=} \frac{2}{\pi} \int_0^{\pi/2} dv \exp \left\{ -z \sin^2(v) \right\} \\ &\stackrel{h=\sin^2 v}{=} \frac{1}{\pi} \int_0^1 \frac{dh}{\sqrt{h(1-h)}} e^{-hz}, \end{aligned} \quad (38)$$

which, following the notation in [JRY08], is the Laplace transform of the variable $\mathbb{G}_{1/2}$ which is arc sine distributed on $[0, 1]$.

$m = 2n$ Lemma 3.1 yields that, $\prod_{k=1}^n \frac{1}{(1+b_k x)}$ is the Laplace transform of an infinitely divisible random variable with Lévy measure:

$$\nu_-(dz) = \frac{dz}{z} \sum_{k=1}^n e^{-z/b_k} = \frac{dz}{z} \sum_{k=1}^n \exp \left\{ -z \sin^2 \left(\frac{\pi}{2} \frac{2k-1}{2n} \right) \right\}. \quad (39)$$

Moreover $\frac{1}{(\prod_{k=1}^n (1+b_k x))^{1/n}}$, for $n \rightarrow \infty$, converges to the Laplace transform of a GGC with Thorin measure density:

$$\mu_-(z) = \mu_+(z). \quad (40)$$

We now express the above results in terms of the Laplace transforms φ_m and $\tilde{\varphi}_m$. Using the following result from [JRY08], p.390, formula (193):

$$\begin{aligned} E \left[\exp \left(-x \Gamma_t \left(\mathbb{G}_{1/2} \right) \right) \right] &= \exp \left\{ -t \int_0^\infty \frac{dz}{z} (1 - e^{-xz}) E \left[\exp \left(-z \mathbb{G}_{1/2} \right) \right] \right\} \\ &= \frac{1}{(\sqrt{1+x} + \sqrt{x})^{2t}} \end{aligned} \quad (41)$$

with $2t = m = \frac{\pi}{2c\varepsilon}$, with c a positive constant, together with (38) and (40), we obtain (18). ■

Remark: The natural question that arises now is whether the results of Proposition 1.2 could be generalized for every $m > 0$ (not necessarily an integer), in other words whether $\varphi_m(x) = \frac{2}{(G_+(x))^m + (G_-(x))^m}$ is the Laplace transform of a generalized Gamma convolution (GGC, see [Bon92] or [JRY08]), that is:

$$\varphi_m(x) = E \left[e^{-x \Gamma_m} \right], \quad (42)$$

with

$$\Gamma_m \stackrel{(law)}{=} \int_0^\infty f_m(s) d\gamma_s, \quad (43)$$

where $f_m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and γ_s is a gamma process.

This conjecture will be investigated in a sequel of this article in [Vak12].

References

- [ADY97] L. Alili, D. Dufresne and M. Yor (1997). Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement Brownien avec drift. In *Exponential Functionals and Principal Values related to Brownian Motion. A collection of research papers; Biblioteca de la Revista Matematica, Ibero-Americana*, ed. M. Yor, 3-14.
- [Ber96] J. Bertoin (1996). Lévy Processes. Cambridge University Press, Cambridge.
- [BPY01] P. Biane, J. Pitman and M. Yor (2001). Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. *Bull. Amer. Math. Soc.*, **38**, 435-465.
- [Bon92] L. Bondesson (1992). Generalized gamma convolutions and related classes of distributions and densities. *Lecture Notes in Statistics*, **76**. Springer-Verlag, New York.
- [Bou83] Ph. Bougerol (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré*, **19**, 369-391.

- [ChY03] L. Chaumont and M. Yor (2003). Exercises in Probability: A Guided Tour from Measure Theory to Random Processes, via Conditioning, Cambridge University Press.
- [ItMK65] K. Itô and H.P. McKean (1965). Diffusion Processes and their Sample Paths. Springer, Berlin Heidelberg New York.
- [JRY08] L.F. James, B. Roynette and M. Yor (2008). Generalized Gamma Convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surveys*, Volume **5** , 346-415.
- [KVA02] E. Koelink, W. Van Assche (eds.) (2002). Orthogonal polynomials and special functions. *Lect. Notes in Mathematics*, Springer-Verlag.
- [Leb72] N.N. Lebedev (1972). Special Functions and their Applications. Revised edition, translated from the Russian and edited by Richard A. Silverman.
- [ReY99] D. Revuz and M. Yor (1999). Continuous Martingales and Brownian Motion. 3rd ed., Springer, Berlin.
- [Riv90] T. J. Rivlin (1990). Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory. John Wiley and Sons, New York.
- [Vak11] S. Vakeroudis (2011). On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity. *Teor. Veroyatnost. i Primenen.-SIAM Theory Probab. Appl.*, **56** (3), 566-591 (in TVP).
- [Vak12] S. Vakeroudis et al. (2012). On Functionals of Brownian motion: Four Essais. Springer Briefs. Editor: M. Yor. In Preparation.
- [VaY11] S. Vakeroudis and M. Yor (2011). Integrability properties and Limit Theorems for the first exit times from a cone of planar Brownian motion. Submitted.
- [Yor80] M. Yor (1980). Loi de l'indice du lacet Brownien et Distribution de Hartman-Watson. *Z. Wahrsch. verw. Gebiete*, **53**, 71-95.
- [Yor01] M. Yor (2001). Exponential Functionals of Brownian Motion and Related Processes. Springer Finance. Springer-Verlag, Berlin.